

Techniques for Calculating two-loop Diagrams

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Abstract

Methods developed by the Bielefeld-DESY-Dubna collaboration in recent years are: DIANA (DIagram ANAlyser), a program to produce “FORM input” for Feynman diagrams, starting from the Feynman rules; methods to calculate scalar diagrams: Taylor expansion in small momenta squared in connection with a mapping and the Padé method to sum the series. Recently program packages for the large mass expansion were written and applied to the $Z \rightarrow b\bar{b}$ decay. Reviews of these activities were presented in the proceedings of the Ustroń '97 and Rheinsberg '98 conferences. Here we concentrate on recent developments in the large mass expansion, applied to the two-loop contribution of the $Z \rightarrow b\bar{b}$ decay in the $m_b = 0$ approximation, taking into account higher order terms of the expansion in M_W^2/m_t^2 .

1 Introduction

The calculation of diagrams with one non-zero external momentum squared (q^2) has wide applications in QED and QCD for both selfenergies and vertices. In these cases also only one non-zero mass enters the problem. In electroweak problems like $Z \rightarrow b\bar{b}$ one has mixing terms between electroweak and strong interactions and due to that different internal masses occur, so that the method of Taylor expansion is getting more difficult to apply. In this case, however, the top quark (m_t) plays a special role and it allows to make the expansion in the large mass. The method is not applicable to arbitrary high q^2 , but as has been demonstrated in ¹, for $q^2 = m_Z^2$ this approach is still reliable. While in ¹ only scalar diagrams have been considered, here we investigate the full decay amplitude. It turns out that the obtained results are simpler for the full process than for scalar diagrams in the following sense: first of all, complicated functions like higher polylogarithms, which show up in the analytic

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evaluation of scalar diagrams, cancel in the full amplitude; furthermore the convergence of the large mass expansion also turns out to be better for the full amplitude than for scalar diagrams. These observations are manifestation of gauge cancellations observed in gauge theories in general. Nevertheless their observation in this special form is surprising!

Due to the fact that the method of mapping and Taylor expansion is quite useful and finds applications by other authors (see e.g. ²), we give a short review here concerning the method and report on recent developments of calculating Taylor coefficients for two-loop diagrams. Then, in the second part, we turn to our main point namely the large mass expansion for the $Z \rightarrow b\bar{b}$ problem and the comparison with the work of³.

2 Expansion of three-point functions in terms of an external momentum squared

Taylor series expansions in terms of one external momentum squared, q^2 say, were considered for selfenergy diagrams in ⁴, Padé approximants were introduced in ⁵ and in Ref. ⁶ it was demonstrated that this approach can be used to calculate Feynman diagrams on their cut by analytic continuation. In the case of a three-point function like $Z \rightarrow b\bar{b}$ in the $m_b = 0$ limit we have for the external b -quark momenta $p_1^2 = p_2^2 = 0$. The expansion of the scalar diagram then looks like

$$C(p_1, p_2) = \sum_{n=0}^{\infty} a_n (p_1 p_2)^n, \quad (1)$$

with $q^2 = (p_1 + p_2)^2$.

For the calculation of the Taylor coefficients in general various procedures have been proposed ^{7,8,9}. These methods are well suited for programming in terms of a formulae manipulating language like FORM ¹⁰. Such programs, however, yield acceptable analytic results only in cases when not too many parameters (like masses) enter the problem. Otherwise numerical methods are needed ¹¹.

In the case of only one non-zero mass and only one external momentum squared, indeed the case with the least nontrivial parameters, for many diagrams analytic expressions for the Taylor coefficients can be obtained. For recent references see ¹².

For the purpose of calculating Feynman diagrams in the kinematical domain of interest it is necessary to calculate them from the Taylor series on their cut. This is performed by analytic continuation in terms of a mapping⁶

Assume, the following Taylor expansion of a scalar diagram or a particular amplitude is given $C(p_1, p_2, \dots) = \sum_{m=0}^{\infty} a_m y^m \equiv f(y)$ and the function on the r.h.s. has a cut for $y \geq y_0$.

The method of evaluation of the original series consists in a first step in a conformal mapping of the cut plane into the unit circle and secondly the reexpansion of the function under consideration into a power series w.r.t. the new conformal variable. We use

$$\omega = \frac{1 - \sqrt{1 - y/y_0}}{1 + \sqrt{1 - y/y_0}}. \quad (2)$$

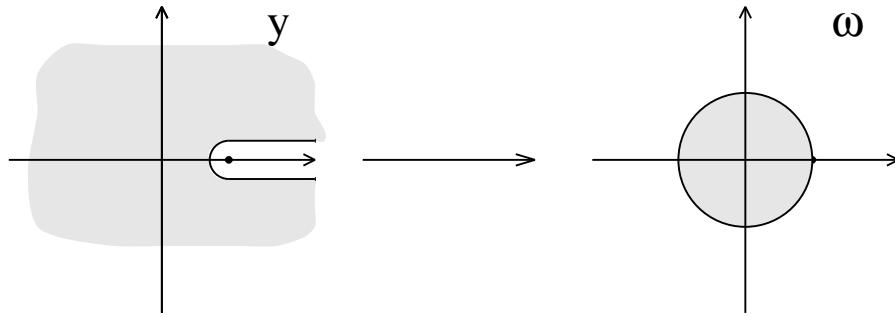


Figure 1: Conformal mapping of the complex y -plane into the ω -plane.

By this conformal transformation, the y -plane, cut from y_0 to $+\infty$, is mapped into the unit circle (see Fig.1) and the cut itself is mapped on its boundary, the upper semicircle corresponding to the upper side of the cut. The origin goes into the point $\omega = 0$.

After conformal transformation it is suggestive to improve the convergence of the new series w.r.t. ω by applying the Padé method^{13,14}. A convenient technique for the evaluation of Padé approximations is the ε -algorithm of¹⁴ which allows one to evaluate the Padé approximants recursively.

Generally speaking, the precision of results with this mapping and Padé is of the order of 3-4 decimals with 30 Taylor coefficients for timelike q^2 values a factor of approximately 100 times the lowest threshold value. For lower q^2 (a few times the threshold value) the precision is of the order of 10 decimals in quite many cases. The precision worsens near second nonzero thresholds.

As a final remark we mention that for diagrams with zero thresholds new techniques have been developed. In fact terms of the form $\ln^m(q^2)$ have to

be factorized, where m is the number of zero thresholds of the diagram. The factors in front are then expanded in terms of Taylor series^{12,15}.

3 Large Mass Expansion (LME)

As mentioned above, for the evaluation of diagrams with several different masses, one of which being large (like the top mass m_t), we use the general method of asymptotic expansion in large masses¹⁶. For a given scalar graph G the expansion in large mass is given by the formula

$$F_G(q, M, m, \varepsilon) \xrightarrow{M \rightarrow \infty} \sum_{\gamma} F_{G/\gamma}(q, m, \varepsilon) \circ T_{q_\gamma, m_\gamma} F_\gamma(q^\gamma, M, m^\gamma, \varepsilon), \quad (3)$$

where γ 's are subgraphs involved in the asymptotic expansion, G/γ denotes shrinking of γ to a point; F_γ is the Feynman integral corresponding to γ ; T_{q_γ, m_γ} is the Taylor operator expanding the integrand in small masses $\{m_\gamma\}$ and external momenta $\{q_\gamma\}$ of the subgraph γ ; \circ stands for the convolution of the subgraph expansion with the integrand $F_{G/\gamma}$. The sum goes over all subgraphs γ which (a) contain all lines with large masses, and (b) are one-particle irreducible w.r.t. light lines.

For the $Z \rightarrow b\bar{b}$ decay we have $q^2 = M_Z^2$ for the on-shell Z 's. Fig.2 shows diagrams with two different masses on virtual lines, one of which a top. W and Z are the gauge bosons with masses M_W and M_Z , respectively; ϕ is the charged would-be Goldstone boson (we use the Feynman gauge); t and b are the t- and b-quarks. Fig. 3 shows subdiagrams needed in the expansion (3).

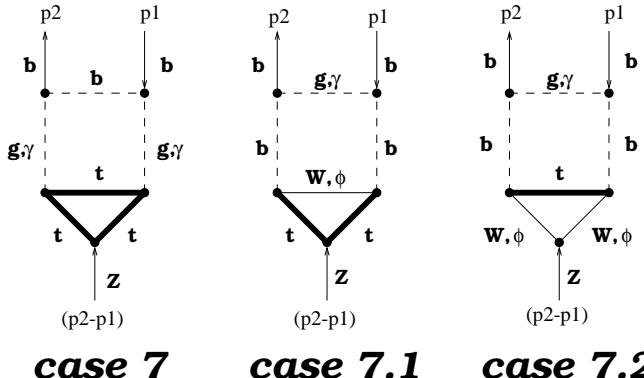


Figure 2: Two-loop diagrams with two different masses on internal lines arising in $Zb\bar{b}$.

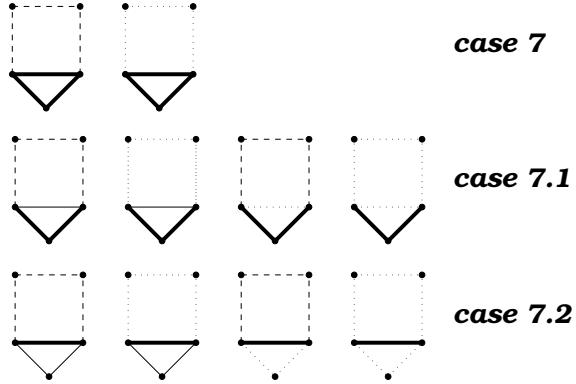


Figure 3: The structure of the LME, see explanations in the text.

Finally the LME of the above diagrams has the following general form

$$F_{\text{as}}^N = \frac{1}{m_t^4} \sum_{n=-1}^N \sum_{i,j=-1; i+j=n}^n \left(\frac{M_W^2}{m_t^2} \right)^i \left(\frac{q^2}{m_t^2} \right)^j \sum_{k=0}^m A_{i,j,k}(q^2, M_W^2, \mu^2) \ln^k \frac{m_t^2}{\mu^2} \quad (4)$$

where m is the highest degree of divergence (ultraviolet, infrared, collinear) in the various contributions to the LME ($m \leq 3$ in the cases considered). M_W^2/m_t^2 and q^2/m_t^2 are considered as small parameters. $A_{i,j,k}$ are in general complicated functions of the arguments, i.e. they may contain logarithms and higher polylogarithms.

In contrary to the work of³ we see no inconveniences in directly applying the above method and did not use any “continued expansion”.

In the following we present some of our results of the LME for the full two-loop $O(\alpha\alpha_s)$ contribution. As in³, we are interested only in the virtual effect of the top quark, which renders the decay of the Z boson into bottom quarks different from the one into other down-type quarks. Therefore in the following the quantity $\Gamma_{b-d}^W = \Gamma_{Z \rightarrow bb}^W - \Gamma_{Z \rightarrow dd}^W$ is considered in two loop order, in which expression the counterterm contributions cancel. The superscript W means that only diagrams with virtual W bosons are included. The other part with the Z exchange makes no discrimination between b - and d -quarks and is calculated in order $\alpha\alpha_s$ in¹⁷. Our result reads

$$\delta\Gamma_{b-d}^{(2),W} = \Gamma^0 \frac{1}{s_\Theta^2} \frac{\alpha}{\pi} \frac{\alpha_s}{\pi} \times$$

$$\begin{aligned}
& \left\{ \frac{M_t^2}{M_W^2} \left[-\frac{1}{32} - \frac{1}{64} \frac{1}{y} + \zeta_2 \left(\frac{1}{16} + \frac{1}{32} \frac{1}{y} \right) \right] \right. \\
& + \left[\frac{61}{288} \frac{1}{y} + \frac{82661}{466560} - \frac{106626671}{204120000} y + \frac{673933}{1458000} y^2 - \frac{12334491149}{16044682500} y^3 + \mathcal{O}(y^4) \right. \\
& + I_1 \left(\frac{1}{96} \frac{1}{y^2} - \frac{5}{192} \frac{1}{y} - \frac{5}{48} - \frac{1}{48} y \right) \\
& + \zeta_2 \left(\frac{173}{1296} + \frac{67}{2592} \frac{1}{y} + \frac{53}{324} y \right) + \zeta_3 \left(-\frac{1}{18} \frac{1}{y} + \frac{7}{90} y - \frac{1}{45} y^2 - \frac{16}{315} y^3 \right) \\
& + L_{tW} \left(-\frac{757}{7776} - \frac{331}{7776} \frac{1}{y} - \frac{95}{3888} y \right) + l_\Theta^2 \left(-\frac{103}{2592} - \frac{1}{81} \frac{1}{y} + \frac{1}{300} y \right. \\
& - \frac{103}{1080} y^2 - \frac{5314}{33075} y^3 \Big) + l_\Theta \left(-\frac{527}{7776} + \frac{11}{288} \frac{1}{y} - \frac{1489}{30375} y + \frac{1081}{9720} y^2 \right. \\
& \left. \left. - \frac{3338578}{10418625} y^3 \right) \right] \\
& + \frac{M_W^2}{M_t^2} \left[\frac{5}{128} \frac{1}{y} - \frac{64847}{1555200} - \frac{636239}{777600} y - \frac{12497}{10800} y^2 \right. \\
& + I_1 \left(-\frac{11}{288} \frac{1}{y^2} - \frac{253}{864} \frac{1}{y} - \frac{11}{48} + \frac{11}{24} y + \frac{11}{108} y^2 \right) \\
& + L_{tW} \left(-\frac{83083}{155520} - \frac{1819}{5184} \frac{1}{y} + \frac{15017}{38880} y + \frac{3977}{38880} y^2 \right) + l_\Theta \left(-\frac{3343}{155520} - \frac{7}{1296} \frac{1}{y} \right. \\
& \left. - \frac{823}{38880} y + \frac{17}{38880} y^2 \right) + \zeta_2 \left(\frac{257}{864} + \frac{175}{864} \frac{1}{y} + \frac{13}{144} y + \frac{11}{18} y^2 \right), \tag{5}
\end{aligned}$$

where Γ^0 is the Born decay rate, $y = M_Z^2/4M_W^2$, $l_\Theta = \ln \cos \Theta_W$, $L_{tW} = \ln(m_t^2/M_W^2)$ and $\zeta_n = \zeta(n)$ the Riemann ζ -function. The following integral is introduced

$$I_n = \frac{1}{2} \frac{(-1)^n}{n!} \int_0^1 \frac{\ln^n(1-ty)}{\sqrt{1-t}} dt. \tag{6}$$

In the above final result enters only I_1 which has the expansion

$$I_1 = \frac{2}{3} y + \frac{4}{15} y^2 + \frac{16}{105} y^3 + \frac{32}{315} y^4 + \frac{256}{3465} y^5 + \frac{512}{9009} y^6 + \frac{2048}{45045} y^7 + \frac{4096}{109395} y^8 + \dots \tag{7}$$

Inserting this expansion in (5), we fully agree with the result of Harlander et.al.³, as far as they have presented their result. Our result is more compact, however, and it is interesting to observe that, while higher polylogarithms occur in the scalar integrals¹, in the full decay amplitude these and the higher I_n

(also, however, expressible in terms of polylogarithms) cancel. The remaining I_1 , expanded above, is merely a logarithm: $I_1 = 2 - a \log((a+1)/(a-1))$, $a = \sqrt{1-1/y}$. Thus we observe that the final result is much simpler than intermediate results from scalar diagrams. Moreover, the convergence of the series in terms of large masses is much better than for the series obtained for scalar diagrams¹, which is demonstrated below.

Our numerical results are as follows: x_1 and x_2 being the 1-loop and 2-loop results, $r = M_W^2/m_t^2$, the large mass expansion is given in the form

$$\delta\Gamma_{b-d}^W = \Gamma^0 \frac{1}{s_\Theta^2} \frac{\alpha}{\pi} \left[x_1 + \frac{\alpha_s}{\pi} x_2 \right], \quad (8)$$

$$x_1 = \left(-\frac{0.1063}{r} \right) + \left(0.2360 - 1.0236r - 1.7492r^2 - 1.4477r^3 - 0.1758r^4 + 1.2997r^5 \right. \\ \left. + 2.1005r^6 + 1.7986r^7 + 0.5692r^8 - 0.9260r^9 - 1.6273r^{10} \right), \quad (9)$$

$$x_2 = \left(\frac{0.2435}{r} \right) + \left(0.7167 - 1.6940r - 4.0920r^2 - 4.7543r^3 - 2.9143r^4 + 0.5688r^5 \right. \\ \left. + 3.7747r^6 + 4.8688r^7 + 3.6932r^8 + 4.5196r^9 + 24.090r^{10} \right). \quad (10)$$

The first terms in x_1, x_2 correspond to the leading m_t^2/M_W^2 . For $m_t = 175\text{GeV}$, $M_W = 80.33\text{GeV}$ and $M_Z = 91.187\text{GeV}$ we obtain

$$\delta\Gamma_{b-d}^W = \Gamma^0 \frac{1}{s_\Theta^2} \frac{\alpha}{\pi} \left[-0.5045 - 0.0704 + \frac{\alpha_s}{\pi} (1.1556 + 0.1285) \right] \quad (11)$$

In each of these terms the leading term and the corrections are given separately. Note that the leading m_t^2/M_W^2 term in order $O(\alpha\alpha_s)$ was obtained earlier in¹⁸.

The observation is that the series for the full amplitude converges like r^n while for the scalar diagram the convergence was like $(4r)^n$. Accordingly the term of order r^4 gives only an error of order 0.5% while for the scalar diagrams the errors were of the order several %¹.

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